Obtaining accurate numerical stiffnesses by routine reduction of the equations for $\underline{K}$ can be a problem for certain ranges of parameters. Reference 3 describes special techniques used to preserve accuracy.

Eigenvalues for Clamped Edges

The iterative analysis procedure of VICONOPT[1] requires the plate stiffnesses to be evaluated at a series of trial values of the eigenvalue [critical buckling load factor or natural frequency of vibration, and not to be confused with the eigenvalues of the $\underline{R}$ matrix of Eq. (11)] that converge to the desired result. For each trial value the analysis[4] requires not only the plate stiffnesses but also the number of eigenvalues exceeded for each individual plate assuming its edges were clamped. The number of eigenvalues exceeded is obtained by dividing the plate width b by 2^n, with n chosen to obtain a plate small enough to guarantee that none of its clamped edge eigenvalues are exceeded. Using this divided plate as a substructure that is repeatedly doubled to return to the original width allows the number of eigenvalues exceeded for clamped edges to be determined. A suitable value of n is determined as follows.

Reference 3 shows that every term of the $\underline{R}$ matrix is proportional to b, and thus the eigenvalues of $\underline{R}$ are proportional to b. Noting that an eigenvalue of $\underline{R}$ equal to π corresponds to buckling or vibration with simply supported edges, choosing n such that all the real eigenvalues of $\underline{R}$ are less than $2^n\pi$ gives $b/2^n$ as the width of a plate for which no buckling or vibration eigenvalues are exceeded if the edges are simply supported, and consequently none are exceeded if the edges are clamped.

Results

Results of the shear deformation analysis applied to the buckling of sandwich plates composed of isotropic layers agree with various published results. Vibration results[3] for a composite cylinder give good agreement with the elasticity solution,[6] indicating that first-order shear deformation theory is adequate for eigenvalue problems. Further results for a typical design problem[3] showed that a stiffened panel with sandwich construction gives significant mass savings compared with solid composite construction. The analysis also indicates the importance of accounting for shear deformations when low mass core materials are used.

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Design Sensitivity Analysis of Structural Frequency Response

T. Ting*
University of Bridgeport, Bridgeport, Connecticut 06601

Introduction

In many modern engineering applications it is desirable to find the effects of design parameter changes on the dynamic response of a system. To date, most research done in this area is on the eigenproblem sensitivity analysis.[1,2] The result of such work is fruitful and almost conclusive. However, there seems to be far less work being done directly on the dynamic response sensitivities, which have even more practical applications. This Note demonstrates the derivation of an improved method for calculating design sensitivities of structural frequency response.

Conventionally, the frequency response of a structure is computed by either the direct or the modal superposition formulation.[3] By differentiating the two types of frequency response equations we can also come up with two kinds of formulations, namely, the direct and the modal formulations.[4] The former is based on the direct frequency response solution and results in an exact calculation of frequency response derivatives for all cases permitted by the analysis formulation. Despite its superb accuracy, the method is accompanied by two drawbacks. The first one is its prohibitive cost of computation when a large system is involved. This is due to its requirement for inverting the system-size matrices as many times as the total number of forcing frequencies involved. The second drawback is its inability in handling modal damping, which can become a vital concern in some applications. The latter takes advantage of the modal formulation, which does not require inverting large matrices. However, it requires computing the eigenvector derivatives, which may sometimes be cost prohibitive. In addition, when there are repeated modes existing among the active modes, the method may fail to obtain the correct result.

An alternative approach is similar to the modal superposition method for the frequency response, which employs a direct modal transformation for the response sensitivities.[5] This method has many practical advantages. It handles a very general class of frequency response problems with great computational efficiency. The primary potential problem associated with this method is the accuracy issue.

This Note provides an improved method to the modal superposition method for calculating frequency response sensitivities. It combines the mode-acceleration method with the Ritz minimization technique to improve the modal approximation accuracy. A similar approach was introduced by Wang for calculating eigenvector derivatives with a very successful result.[6] In addition, an iteration scheme is used to further enhance the result. Such an approach has also been used for eigenvector derivatives.[7,8]

Theoretical Background

Frequency Response

For a harmonic excitation, the structural response is expected to be also harmonic with the same frequency. The

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*Associate Professor, Mechanical Engineering. Member AIAA.

dynamic equilibrium equation associated with any excitation frequency ω_i can be expressed as follows:

$$D_i u_i = p_i \tag{1}$$

where u_i is the response vector with respect to frequency ω_i, p_i the force vector, and

$$D_i = -\omega_i^2 M + i\omega_i B + K \tag{2}$$

with M, B, and K being the mass, damping, and stiffness matrices, respectively, of the structure. The response vector u_i may be calculated by directly solving Eq. (1), which involves decomposition of D_i and then forward/backward substitution for each excitation frequency. Although the direct approach gives the exact solution to Eq. (1), it becomes computationally prohibitive when the number of excitation frequencies is large and the system size is also large.

As a result of many unsolved modeling difficulties, such as damping and dynamic boundary conditions, the exact analytical solution may not closely predict the true response. Hence, for most practical applications of dynamic response information, a fair approximation to the analytically exact value may work as effectively. Modal approximation is normally the most widely adopted method for approximating frequency responses. It is assumed that response due to forces of certain frequency will have large participation factors for nearby natural modes. This means that if a subset of modal vectors are used to form a basis for approximating a frequency response vector of frequency ω_i, the quality of the approximation is normally determined by how many neighboring modal vectors are included in the subset.

Let $\bar{\Phi}$ be a subset of modal vectors; the modal approximation of u_i can then be written as:

$$\tilde{u}_i = \bar{\Phi}\xi_i \tag{3}$$

By substituting $\tilde{u}_i$ for u_i in Eq. (1), then premultiplying by $\bar{\Phi}^T$, Eq. (1) becomes

$$\bar{\Phi}^T D_i \bar{\Phi}\xi_i = -\bar{\Phi}^T p_i \tag{4}$$

This is the reduced frequency response equation which together with Eq. (3) can achieve a reasonable and economical approximation of u_i if the associated natural frequency of $\bar{\Phi}$ contains the neighborhood of ω_i.

Frequency Response Sensitivity

Now consider the direct formulation for calculating frequency response derivatives as

$$D_i u_{i,j} = -D_{i,j} u_i \tag{5}$$

where, for example, $D_{i,j} = (\partial/\partial x_j) D_i$. Employing a modal transformation

$$u_{i,j} = \Phi \eta_{ij} \tag{6}$$

where Φ is the modal matrix and η_{ij} are the modal factors for $u_{i,j}$. Substituting Eq. (6) into Eq. (5) for $u_{i,j}$ and premultiplying by Φ^T, we obtain

$$\Phi^T D_i \Phi \eta_{ij} = -\Phi^T D_{i,j} u_i \tag{7}$$

If a truncated modal matrix $\bar{\Phi}$ is used in Eq. (6), then

$$\tilde{u}_{i,j} = \bar{\Phi}\tilde{\eta}_{ij} \tag{8}$$

where $\tilde{u}_{i,j}$ is an approximation for $u_{i,j}$ and $\tilde{\eta}_{ij}$ is determined by solving the reduced equation of

$$\bar{\Phi}^T D_i \bar{\Phi}\tilde{\eta}_{ij} = -\bar{\Phi}^T D_{i,j} u_i \tag{9}$$

Such an approximation should possess the same degree of accuracy as using the truncated modal response solution in practice. The accuracy of the approximation is deteriorated when the response u_i on the right-hand side of Eq. (9) is also a modal approximation. This sometimes can lead to an unacceptable result.

Improved Method

However, there are a few ways to improve the quality of $\tilde{u}_{i,j}$ as an approximation for $u_{i,j}$. The most basic approach is to increase the number of modes in $\bar{\Phi}$ or use the mode acceleration method. Such methods are easy to apply but may achieve little improvements or fail to produce results, and sometimes the cost is too high. An improved method has been sought to be a general as well as an efficient approach to enhance the modal approximation of Eqs. (8) and (9).

Based on the fundamental sensitivity equation, i.e., Eq. (5), a static equation can be written as

$$K^R u_{i,j} = -D_{i,j} u_i - (-\omega_i^2 M + i\omega B + iK^I) u_{i,j} \tag{10}$$

where a complex stiffness matrix is assumed and K^R and K^I are the real and imaginary part of K, respectively. This shows the mode acceleration method. Now, let

$$f_{ij} = Re\{-D_{i,j} u_i - (-\omega_i^2 M + i\omega B + iK^I) u_{i,j}\} \tag{11}$$

and

$$K^R w_{ij} = f_{ij} \tag{12}$$

Since K^R is frequently singular in most aerospace and automobile applications, it is desirable to use a shifted formulation such as

$$(K^R + \sigma M)\bar{w}_{ij} = f_{ij} \tag{13}$$

where $\bar{w}_{ij}$ can be considered as the static mode associated with f_{ij}. To make the static mode independent from all modes in $\bar{\Phi}$, we can apply a modal filtration to yield

$$(K^R + \sigma M)\hat{w}_{ij} = f_{ij} - \bar{\Phi}(\bar{\Lambda} + \sigma I)^{-1}\bar{\Phi}^T f_{ij} \tag{14}$$

Alternatively, the modal filtration can also be achieved as follows:

$$\hat{w}_{ij} = (I - \overline{\Phi\Phi}^T M)\bar{w}_{ij} \tag{15}$$

where $\bar{w}_{ij}$ is determined from Eq. (13). Either Eq. (14) or Eq. (15) may be used to calculate a set of static modes which are all M-orthogonal to modes in $\bar{\Phi}$, i.e., for any $\hat{w}_{ij}$,

$$\bar{\Phi}^T M \hat{w}_{ij} = 0 \tag{16}$$

These static modes can be used as additional Ritz vectors to improve the modal approximation of Eq. (3). They can be evaluated simultaneously and thus provide a very economical way for obtaining them.

Assume that matrices $\hat{W}$ and F represent a collection of $\hat{w}_{ij}$ and f_{ij}, respectively, with the selected combination of design variable and excitation frequency groups. Then a Ritz minimization can be given as

$$\hat{W}^T D_i \hat{W}\hat{\eta}_{ij} = -\hat{W}^T D_{i,j} u_i \tag{17}$$

which is similar to Eq. (9). Solve for $\hat{\eta}_{ij}$ in Eq. (17) and then compute

$$\hat{u}_{i,j} = \hat{W}\hat{\eta}_{ij} \tag{18}$$

The improved approximation becomes

$$\bar{u}_{i,j} = \tilde{u}_{i,j} + \hat{u}_{i,j} \tag{19}$$

Iterative Scheme

The improved method just described provides a better approximation to $u_{i,j}$ than the conventional modal method does. Since Eq. (10) requires an estimate of $u_{i,j}$ on the right-hand side of the equation, the accuracy of the approximation is heavily dependent on this estimate. Typically, we shall use the modal approximation given in Eq. (8) as the initial estimate. A recurrence relation based on equations derived can thus be established, and the approximation can be further improved by an iterative scheme.

A model iterative scheme for perfecting the approximation to $u_{i,j}$ can be summarized in the following:

1) Set $k = 0$ and $u_{ij}^{(0)} = \tilde{u}_{ij}$ of Eq. (8).

2) $k = k + 1$.

3) Compute static modes simultaneously based on the recurrence form of Eq. (13), i.e.,

$$(K^R + \sigma M)\bar{W}^{(k)} = F^{(k-1)} \tag{20}$$

where columns of $F^{(k+1)}$ are computed using Eq. (11) with $u_{i,j}^{(k-1)}$ in the place of $u_{i,j}$.

4) Apply modal filtration to yield

$$\hat{W}_{ij}^{(k)} = (I - \overline{\Phi\Phi}^T M)\bar{W}_{ij}^{(k)} \tag{21}$$

5) Solve the Ritz minimization problem

$$\hat{W}^{T(k)} D_i \hat{W}^{(k)} \hat{\eta}_{ij}^{(k)} = -\hat{W}^{T(k)} D_{i,j} u_i \tag{22}$$

6) Update the approximation

$$u_{i,j}^{(k)} = \tilde{u}_{i,j} + \hat{W}^{(k)} \hat{\eta}_{ij}^{(k)} \tag{23}$$

7) Go to Step 2 if needed.

8) Stop the process.

The computational cost in the preceding algorithm can be further reduced by simultaneous calculation of all of the required $u_{i,j}$. The selection of the set of static modes is also important. Conventionally, the normal modes used in the modal approximation are all of the lower modes spanning from zero to above the maximal excitation frequency, say ω_{max}. This suggests that additional Ritz vectors rich in modes of frequencies higher than ω_{max} are the most needed for improving the approximation accuracy. Keeping in this line, one may select only those f_{ij} which are related to ω_{max} for generating the static modes to be used as the additional Ritz vectors in the iterative scheme. This not only minimizes the computational cost, it also cuts down the dependency problem among all of the Ritz vectors.

Summary

Expressions and an iterative scheme for improving the present methods in calculating frequency response sensitivities have been derived. These results have vast potential in application to dynamic response problems. This method provides a relatively economical way to yield accurate design sensitivity data for frequency response of a general structural system. Further work in providing comparison to the previously published methods is underway and will be submitted for future publication.

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